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Computational Mechanics

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- Computational Mechanics
- Numerical simulation process
- Finite Difference Method
- Variational approach
- Finite Element Method
- Finite Volume Method
- Galerkin Discontinuous Finite Element Method

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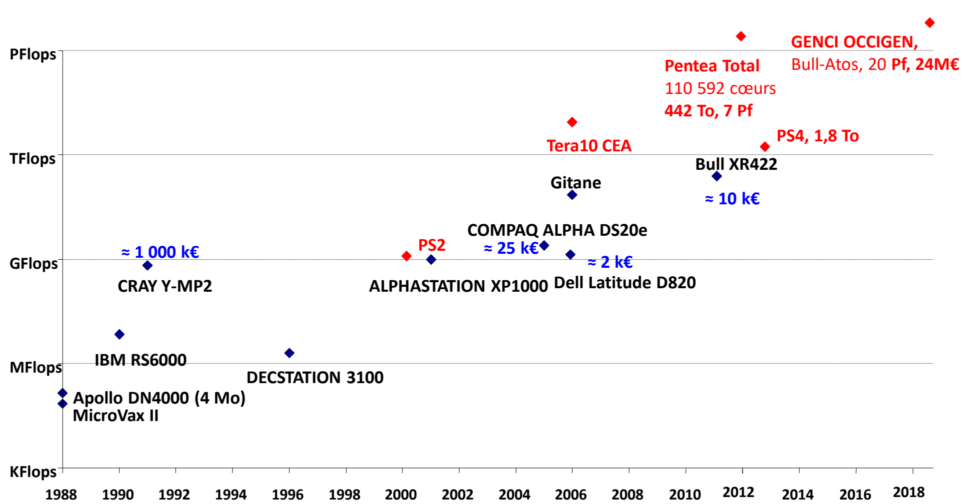
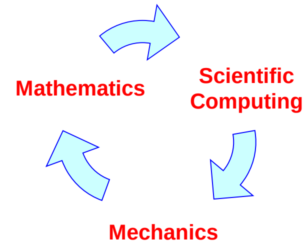
Computational Mechanics

Evolution of computeurs



Computational Mechanics

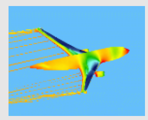
Evolution of computational mechanics



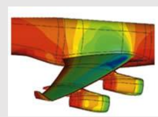
Aim:
- Reproduce to explain the inaccessible
- Plan before you realize

Replace
Theory-Experiment
by
Theory-computation-Experiment

Evolution:
- Multiphysics
- Multiscale
- High Performance Computing



aerodynamic



aeroelasticity

Structural analysis

Sources : engineering.swan
ONERA & Lecture NF04 UTC

Conception — Simulation — Experiment — Production

Computational Mechanics

Continuum mechanics

Conservation of mass

$$\frac{d\rho}{dt}+\rho\,\mathrm{div}\vec{v}=0$$

or

$$\frac{\partial\rho}{\partial t}+\mathrm{div}(\rho\vec{v})=0$$

Balance of linear momentum

$$\begin{cases}\mathrm{div}\overline{\overline{\sigma}}+\vec{f}=\rho\vec{\gamma} & \text{dans }\Omega\\ \overline{\overline{\sigma}}\vec{n}=\vec{F} & \text{sur }\partial\Omega\end{cases}$$

Balance of angular momentum

$$\overline{\overline{\sigma}}=\overline{\overline{\sigma}}^{T}$$

Conservation of energy

$$\rho\frac{de}{dt}-\overline{\overline{\sigma}}:\overline{\overline{D}}=r-\mathrm{div}\vec{q}$$

- $[m]$ $\vec{x}$: actual position
- $[s]$ t : time
- $(\vec{x},t)$: Eulerian variables
- $[kg/m^3]$ $\rho(\vec{x},t)$: density
- $[m/s]$ $\vec{v}(\vec{x},t)$: velocity
- $[N/m^3]$ $\vec{f}$: body force per unit volume acting in Ω
- $[N/m^2]$ $\vec{F}$: contact force per unit surface acting on $\partial\Omega$
- $[-]$ $\vec{n}$: outward normal vector of $\partial\Omega$
- $[m/s^2]$ $\vec{\gamma}$: acceleration
- $[N/m^2]$ $\overline{\overline{\sigma}}(\vec{x},t)$: Second order Cauchy stress tensor
- $[W/kg]$ e : specific internal energy
- $[s^{-1}]$ $\overline{\overline{D}}$: strain-rate (or stretching) tensor
- $[W/m^2]$ $\vec{q}$: heat flux
- $[W/m^3]$ r : heat supply density

Material derivative

$$\frac{d}{dt}=\frac{\partial}{\partial t}+(\vec{v}\cdot\nabla)$$

$$\vec{\gamma}(\vec{x},t)=\frac{d\vec{v}}{dt}=\frac{\partial\vec{v}}{\partial t}+(\vec{v}\cdot\nabla)\vec{v}$$

$$\overline{\overline{D}}=\frac{1}{2}(\nabla^{T}\vec{v}+\nabla\vec{v})$$

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Example: Wave breaking

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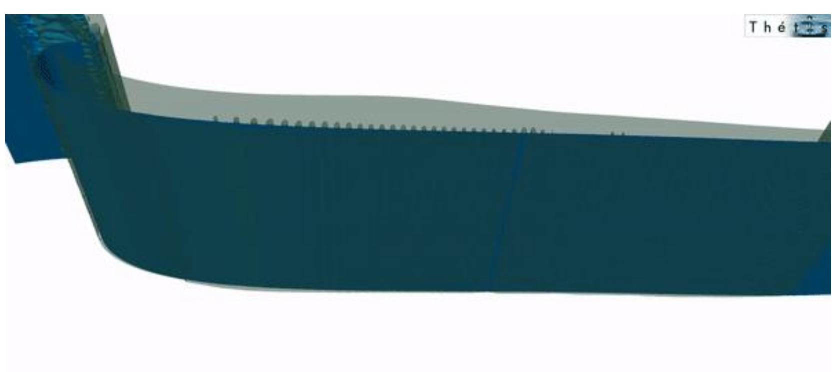
Computational Mechanics

Example: Wave breaking, full Navier-Stokes Air-Water simulation



P. Lubin, S. Glockner I2M Bordeaux, France, software Thétis

80 Millions cells on 1024 cores

800 Millions cells on 4096 cores

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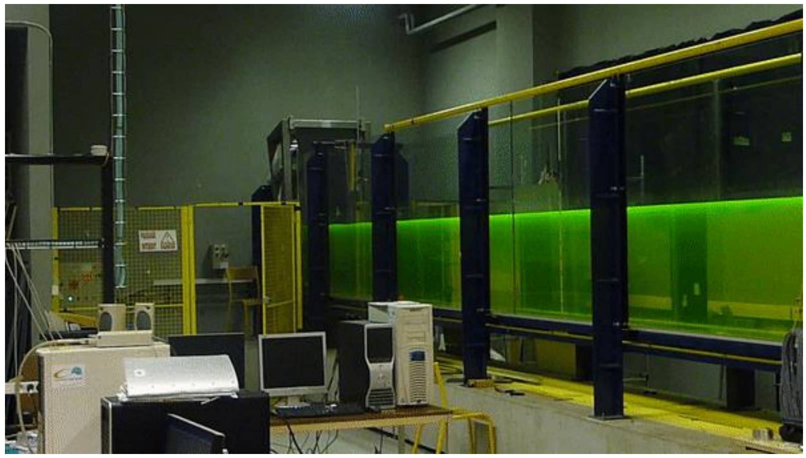
Computational Mechanics

Example: Wave propagation, wave breaking ….. experiment



Olivier Kimmoun, ECM, Marseille France, workshop B'Waves 2014 Bordeaux

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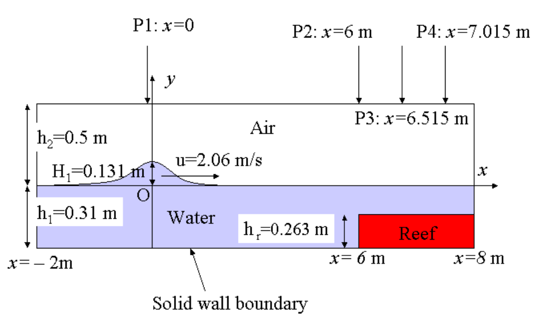
Computational Mechanics

Example: : Wave propagation, wave breaking ….. experiment



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Example: Wave propagation, wave breaking ….. Air-Water Euler model (Finite Volume)

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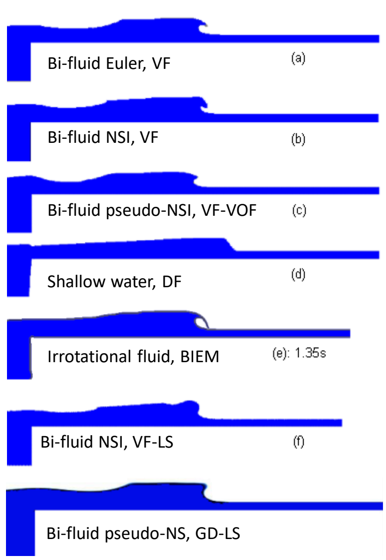
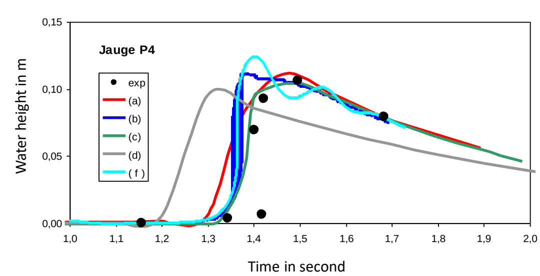
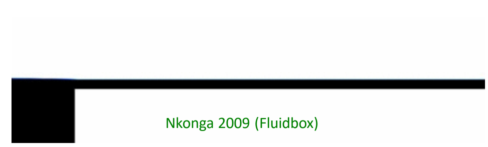
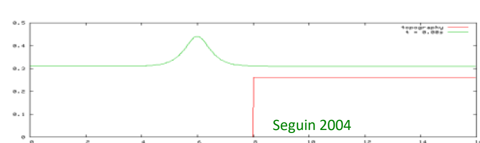
Computational Mechanics

Confrontation of several CFD codes









P. Helluy, F. Golay, J.P. Caltagirone, P. Lubin, S. Vincent, D. Drevard, R. Marcer, P. Fraunie, N. Seguin, S. Grilli, A.N. Lesage, A. Dervieux, O.Allain, “Numerical simulation of wave breaking”, M2AN, Vol.39 n°3, pp 591-607, **2005**

B. Braconnier, J-J Hu, Y-Y Niu, B. Nkonga, K-M Shyue , Numerical simulations of low Mach compressible two-phase flows: Preliminary assessment of some basic solution techniques , ESAIM: Proc., Vol. 28, pp. 117-134, 2009.

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Example, metal forming using finite elements (Forge3D – Cemef)

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Computational Mechanics

Example of computation on GPU (DEM+Cuda)

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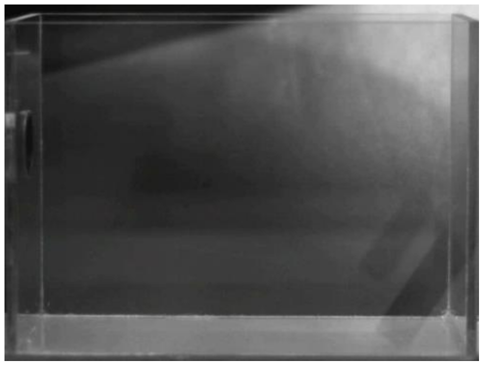
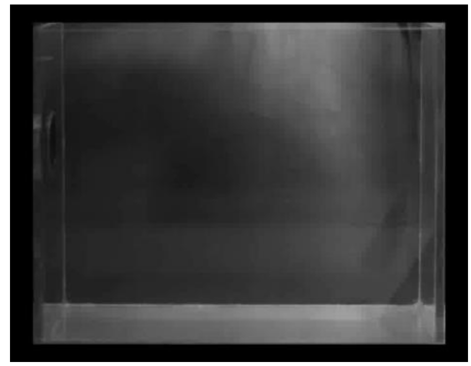
Computational Mechanics

Realistic image rendering example applied to fluid mechanics



Experiment



Numerical simulation:
FEM/VMS + LES +
Convected Level Set…
+ blender

Elie Hachem, Thierry Coupez, Cemef, code CimLib, anisotropic adaptive mesh refinement technic

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Computational Mechanics

Overview of numerical approximation of partial differential equations

- Finite Difference Method (FDM)
- Finite Element Method (FEM)
- Finite Volume Method (FVM)
- Spectral Method
- Boundary Element Method (BEM)
- Lattice Boltzmann (LB)
- Smooth Particle Hydrodynamics (SPH)
- Disrete Element Method (DEM)
- Many others …

Finite Difference Method (FDM)
The oldest method, used for many physics
- Easy to implement
- Easy to make higher order
- Only applicable on structured grids
- Does not fit the geometry

Finite Element Method (FEM)
Usually used in structure mechanics
- Any order of accuracy
- Based on variational methods
- Applicable on unstructured/complex grids
- Naturally implicit, well-adapted to non-linear behaviour
- Excellent for diffusion dominated problem
- Naturally implicit, need sparse linear solver (can be expensive for large systems)
- Not conservative
- More complex

Finite Volume Method (FVM)
Usually used in fluid mechanics
- Applicable on unstructured/complex grids
- Naturally conservative, capture discontinuities
- Easy explicit formulation (parallel computation)
- Difficult to devise stable higher order scheme
- Not very accurate for diffusion dominated problem
- More complex

- Computational Mechanics
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Numerical simulation process

What does computing engineer mean?

system analysis, determination of the main phenomena, ...
Fluid flows, wave propagation, wave breaking, impact, fluid-structure interaction, structural analysis, …

- Computational Mechanics
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Numerical simulation process

What does computing engineer mean?

System analysis, modelization ,… Fluid flow

Mechanical / Mathematical Modelization

?

Model 1: Shallow water

$$\begin{cases} \frac{\partial h}{\partial t} + \mathrm{div}(h\vec{u}) = 0 \\ \frac{\partial h\vec{u}}{\partial t} + \mathrm{div}(h\vec{u}\otimes\vec{u} + g\frac{h^2}{2}\bar{\bar{I}}) = -\rho h \nabla Z + ... \end{cases}$$

Model 2: Navier Stokes

$$\begin{cases} \mathrm{div}(\vec{v}) = 0 \\ \frac{\partial \vec{v}}{\partial t} + \nabla\vec{v}.\vec{v} = -\frac{1}{\rho}\nabla p + \nu\Delta\vec{v} \end{cases}$$

Model 3: Euler two-phase

$$\begin{cases} \frac{\partial \rho}{\partial t} + \mathrm{div}(\rho\vec{u}) = 0 \\ \frac{\partial \rho\vec{u}}{\partial t} + \mathrm{div}(\rho\vec{u}\otimes\vec{u} + p\bar{\bar{I}}) = \rho\vec{g} \\ \frac{\partial \varphi}{\partial t} + \vec{u}\cdot\nabla\varphi = 0 \end{cases}$$

Model 4: .....

- Computational Mechanics
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Numerical simulation process

What does computing engineer mean?

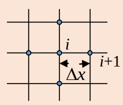
System analysis, modelization ,... Fluid Flow

Mechanical / Mathematical Modelization : Bifluid Euler

Numerical formulation, discrete approach

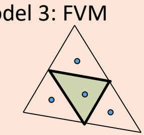
?

Model 1: FDM

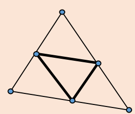


Model 4: SPH

Model 3: FVM



Model 2: FEM



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Numerical simulation process

What does computing engineer mean?

System analysis, modelization ,... -> Fluid flow

Mechanical / Mathematical Modelization -> Bifluid Euler

Numerical formulation, discrete approach -> FVM

Numerical method, solver

Spatial discretisation: order, gradient, …
Time discretization: explicit, implicit, …
Solver: direct, iterative, …
Domain decompositon, mesh refinement, …
Numerical improvement, …

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Numerical simulation process

What does computing engineer mean?

System analysis, modelization ,… -> Fluid flow 

↓

Mechanical / Mathematical Modelization -> Bifluid Euler

↓

Numerical formulation, discrete approach -> FVM

↓

Numerical method, solver -> Explicit second order parallel

↓

Software implementation process

Programmation: C, fortran90, C++, …
Parallel: Cuda, MPI, …
Computer-aided software engineering, Scientific programming, versioning, …

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Numerical simulation process

What does computing engineer mean?

System analysis, modelization ,… -> Fluid flow 

↓

Mechanical / Mathematical Modelization -> Bifluid Euler

↓

Numerical formulation, discrete approach -> FVM

↓

Numerical method, solver -> Explicit second order parallel

↓

Software implementation process

↓

Computation implementation: geometry, mesh, initial conditions, boundary conditions, …

Preprocessing: can be very time-consuming

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Numerical simulation process

What does computing engineer mean?

System analysis, modelization ,... -> Fluid flow 

↓

Mechanical / Mathematical Modelization -> Bifluid Euler

↓

Numerical formulation, discrete approach -> FVM

↓

Numerical method, solver -> Explicit second order parallel

↓

Software implementation process

↓

Computation implementation: geometry, mesh, initial conditions, boundary conditions, …

↓

Run !

HPC: High Performance Computing

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System analysis, modelization ,... -> Fluid flow 

↓

Mechanical / Mathematical Modelization -> Bifluid Euler

↓

Numerical formulation, discrete approach -> FVM

↓

Numerical method, solver -> Explicit second order parallel

↓

Software implementation process

↓

Computation implementation: geometry, mesh, initial conditions, boundary conditions, …

↓

Run !

↓

Very critical analysis of the results

- Computational Mechanics
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Numerical simulation process

What does computing engineer mean?

CFD = Computational Fluid Dynamics

CFD ≠ Color Fluid Dynamics (G. Allaire ?)

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System analysis, modelization ,... 

↓

Mechanical / Mathematical Modelization

↓

Numerical formulation, discrete approach

↓

Numerical method, solver

↓

Software implementation process

↓

Computation implementation: geometry, mesh, initial conditions, boundary conditions, ...

↓

Run !

↓

Very critical analysis of the results

- Computational Mechanics
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Numerical simulation process

What does computing engineer mean?

System analysis, modelization ,…

Software:
- Mechanical / Mathematical Modelization
- Numerical formulation, discrete approach
- Numerical method, solver
- Software implementation process

Computation implementation: geometry, mesh, initial conditions, boundary conditions, …

Run !

Very critical analysis of the results

- Computational Mechanics
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Numerical simulation process

What does computing engineer mean?

System analysis, modelization ,…

Software:
- Mechanical / Mathematical Modelization
- Numerical formulation, discrete approach
- Numerical method, solver
- Software implementation process

Computation implementation: geometry, mesh, initial conditions, boundary conditions, …

Run !

Very critical analysis of the results

Bad engineer

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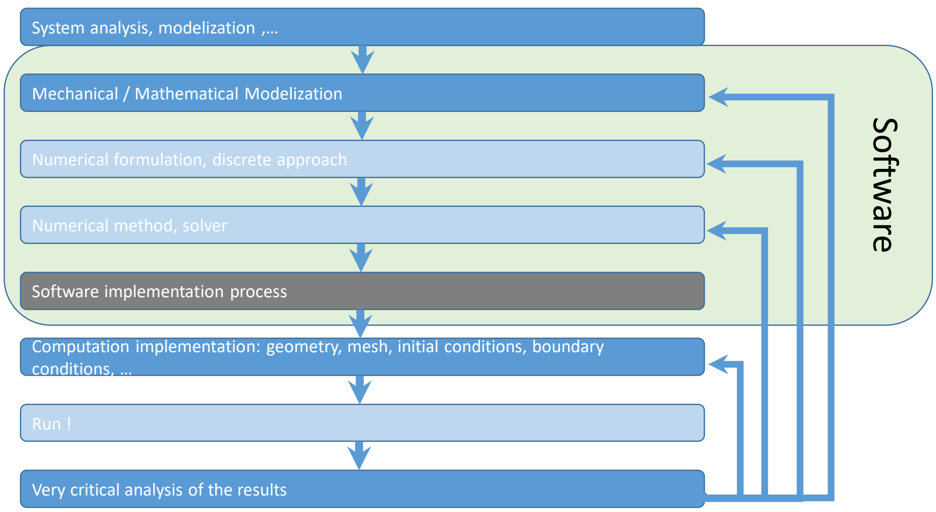
Numerical simulation process

What does computing engineer mean?



Most engineer

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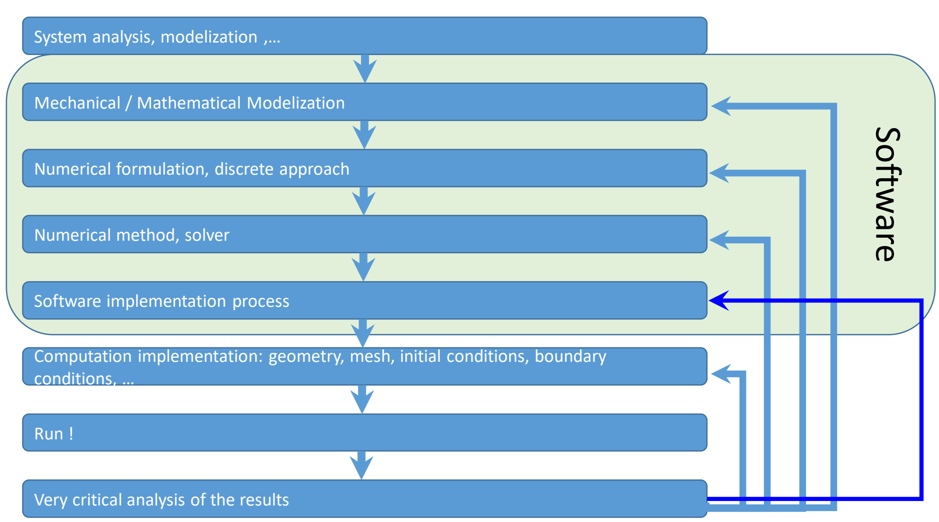
Numerical simulation process

What does computing engineer mean?



Engineer in computational Mechanics

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Finite Difference Method

Taylor's Theorem

Let u be an n-times differentiable function on an open interval containing the points x+h and x, then :

$$u(x+h)=u(x)+hu'(x)+\frac{h^2}{2!}u^{(2)}(x)+\frac{h^3}{3!}u^{(3)}(x)+\cdots+\frac{h^{n-1}}{(n-1)!}u^{(n-1)}(x)+\bigcirc(h^n)$$

Where $\bigcirc(h^n)=\frac{h^n}{n!}u^n(c)$ and c is between x and x+h

Finite difference approximation of the first derivative of u

Forward first order finite difference approximation of u'

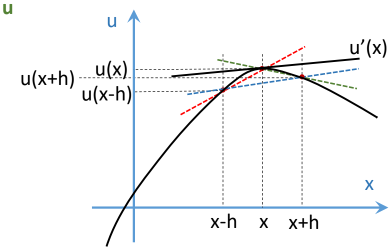
$$u(x+h)=u(x)+hu'(x)+\bigcirc(h^2) \qquad u'(x)=\frac{u(x+h)-u(x)}{h}+\bigcirc(h)$$

Backward first order finite difference approximation of u'

$$u(x-h)=u(x)-hu'(x)+\bigcirc(h^2) \qquad u'(x)=\frac{u(x)-u(x-h)}{h}+\bigcirc(h)$$

Centered first order finite difference approximation of u'

$$u'(x)=\frac{u(x+h)-u(x-h)}{2h}+\bigcirc(h)$$



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Finite Difference Method

Finite difference approximation of the second derivative of u

$$u(x+h)=u(x)+hu'(x)+\frac{h^2}{2}u''(x)+\frac{h^3}{6}u^{(3)}(x)+\bigcirc(h^4)$$

$$u(x-h)=u(x)-hu'(x)+\frac{h^2}{2}u''(x)-\frac{h^3}{6}u^{(3)}(x)+\bigcirc(h^4)$$

$$u(x+h)+u(x-h)=2u(x)+h^2u''(x)+\bigcirc(h^4)$$

$$u''(x)=\frac{u(x+h)-2u(x)+u(x-h)}{h^2}+\bigcirc(h^2)$$

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Finite Difference Method — Elliptic equation

1D discretization of the Laplacian operator

$$\Delta u = 0$$

$$\frac{\partial^2 u}{\partial x^2} = 0$$

$$h = \frac{L}{n+1}$$

Notation $u(x_i) = u_i$

$$\forall i = 1,n \quad u''(x_i) \approx \frac{u_{i+1} - 2u_i + u_{i-1}}{h^2} = 0$$

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Finite Difference Method — Elliptic equation

2D discretization of the Laplacian operator

$$u(x+h_x, y+h_y) = \sum_{0 \le j+k \le p} \frac{h_x^j h_y^k}{j!k!} \frac{\partial^{j+k} u(x,y)}{\partial x^j \partial y^k}$$

$$\Delta u(x,y) = 0$$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

$$u(x+h_x, y) = u(x,y) + h_x \frac{\partial u(x,y)}{\partial x} + \frac{h_x^2}{2} \frac{\partial^2 u(x,y)}{\partial x^2} + \frac{h_x^3}{6} \frac{\partial^3 u(x,y)}{\partial x^3} + O(h_x^4)$$

$$u(x-h_x, y) = u(x,y) - h_x \frac{\partial u(x,y)}{\partial x} + \frac{h_x^2}{2} \frac{\partial^2 u(x,y)}{\partial x^2} - \frac{h_x^3}{6} \frac{\partial^3 u(x,y)}{\partial x^3} + O(h_x^4)$$

$$u(x, y+h_y) = u(x,y) + h_y \frac{\partial u(x,y)}{\partial y} + \frac{h_y^2}{2} \frac{\partial^2 u(x,y)}{\partial y^2} + \frac{h_y^3}{6} \frac{\partial^3 u(x,y)}{\partial y^3} + O(h_y^4)$$

$$u(x, y-h_y) = u(x,y) - h_y \frac{\partial u(x,y)}{\partial y} + \frac{h_y^2}{2} \frac{\partial^2 u(x,y)}{\partial y^2} - \frac{h_y^3}{6} \frac{\partial^3 u(x,y)}{\partial y^3} + O(h_y^4)$$

$$\frac{\partial^2 u(x,y)}{\partial x^2} + \frac{\partial^2 u(x,y)}{\partial y^2} = \frac{u(x+h_x,y) - 2u(x,y) + u(x-h_x,y)}{h_x^2} + \frac{u(x,y+h_y) - 2u(x,y) + u(x,y-h_y)}{h_y^2} + O(h_x^2) + O(h_y^2)$$

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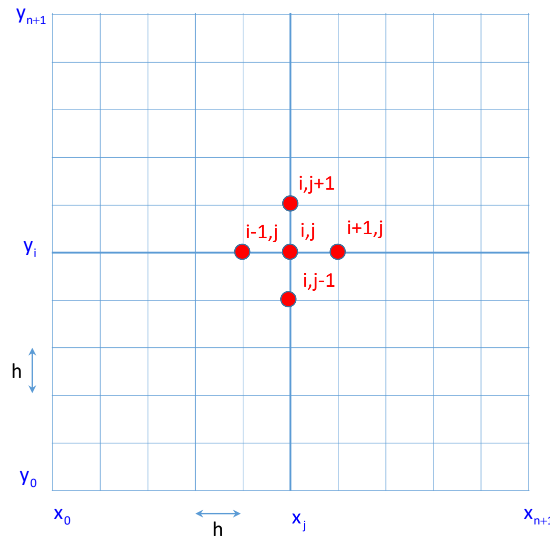
Finite Difference Method Elliptic equation

2D five points stencil

Notation $u(x_i, y_j) = u_{i,j}$

$\forall i,j = 1,n$

$$\Delta u(x_i, y_j) \approx \frac{u_{i+1,j} + u_{i,j+1} - 4u_{i,j} + u_{i-1,j} + u_{i,j-1}}{h^2} = 0$$



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Finite Difference Method Parabolic equation

1D discretization

$$\begin{cases} \dfrac{\partial u}{\partial t}(x,t) - \lambda \Delta u = 0 \\ x(x,0) = u_0(x) \end{cases}$$

Notation $u(x_i, t_n) = u_i^n$

$$\frac{\partial u}{\partial t}(x_i, t_n) \approx \frac{u_i^{n+1} - u_i^n}{\delta t}$$ First order approximation

Time discretisation: explicit Euler scheme

$$\frac{u_i^{n+1} - u_i^n}{\delta t} = \lambda \frac{\partial^2 u}{\partial x^2}(x_i, t_n) \approx \lambda \frac{u_{i+1}^n - 2u_i^n + u_{i-1}^n}{h^2} \rightarrow u_i^{n+1} = u_i^n + \frac{\lambda \delta t}{h^2}\left(u_{i+1}^n - 2u_i^n + u_{i-1}^n\right)$$

Time discretisation: implicit Euler scheme

$$\frac{u_i^{n+1} - u_i^n}{\delta t} = \lambda \frac{\partial^2 u}{\partial x^2}(x_i, t_{n+1}) \approx \lambda \frac{u_{i+1}^{n+1} - 2u_i^{n+1} + u_{i-1}^{n+1}}{h^2} \rightarrow -u_{i+1}^{n+1} + \left(\frac{h^2}{\lambda \delta t} + 2\right) u_i^{n+1} - u_{i-1}^{n+1} = \frac{h^2}{\lambda \delta t} u_i^n$$

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Finite Difference Method   Parabolic equation

Time discretisation: semi-implicit scheme

$$\frac{u_i^{n+1}-u_i^n}{\delta t} = \theta\lambda \frac{\partial^2 u}{\partial x^2}(x_i,t_{n+1}) + (1-\theta)\frac{\partial^2 u}{\partial x^2}(x_i,t_n) \approx \theta\lambda \frac{u_{i+1}^{n+1}-2u_i^{n+1}+u_{i-1}^{n+1}}{h^2} + (1-\theta)\lambda\frac{u_{i+1}^n-2u_i^n+u_{i-1}^n}{h^2}$$

$$A = \begin{bmatrix} 2 & -1 & 0 & \cdots & \cdots & \cdots & 0 \\ -1 & 2 & -1 & 0 & & & 0 \\ 0 & \ddots & \ddots & \ddots & \ddots & & \vdots \\ \vdots & \ddots & -1 & 2 & -1 & \ddots & \vdots \\ \vdots & & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & & & 0 & -1 & 2 & -1 \\ 0 & \cdots & \cdots & \cdots & 0 & -1 & 2 \end{bmatrix} \qquad U^n = \begin{Bmatrix} u_1^n \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ u_N^n \end{Bmatrix} \qquad \left(I_d + \frac{\theta\lambda\delta t}{h^2}A\right)U^{n+1} = \left(I_d - \frac{(1-\theta)\lambda\delta t}{h^2}A\right)U^n$$

θ=0 explicit scheme: Forward Euler	Conditionally stable
θ=1 implicit scheme: Backward Euler	Unconditionally stable
θ=0,5 semi implicit scheme: Crank-Nicholson	Unconditionally stable for θ≥0,5

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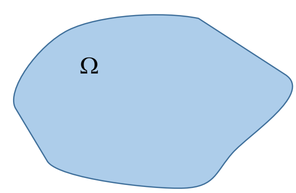
Variational approach

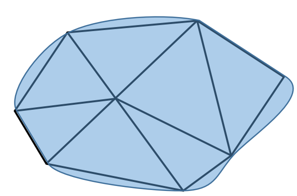
Principle

$$\frac{d\vec{w}}{dt} + \mathrm{div}\overline{\overline{F}}(w) = \vec{S} \quad \text{in } \Omega$$

$$\int_\Omega \vec{\varphi}\cdot\left(\frac{d\vec{w}}{dt} + \mathrm{div}\overline{\overline{F}}\right)dv = \int_\Omega \vec{\varphi}\cdot\vec{S}\,dv \quad \forall\vec{\varphi}\ \ldots$$



The computational domain Ω is split in several cells or elements



$$\int_\Omega \ldots\, dv = \sum_e \int_{\Omega_e} \ldots\, dv$$

$$\sum_e \int_{\Omega_e} \vec{\varphi}\cdot\frac{d\vec{w}}{dt}\,dv + \sum_e \int_{\Omega_e} \vec{\varphi}\cdot\mathrm{div}\overline{\overline{F}}\,dv = \sum_e \int_{\Omega_e} \vec{\varphi}\cdot\vec{S}\,dv$$

$$\sum_e \int_{\Omega_e} \vec{\varphi}\cdot\frac{d\vec{w}}{dt}\,dv - \sum_e \int_{\Omega_e} \nabla\vec{\varphi}:\overline{\overline{F}}^T\,dv + \sum_e \int_{\Omega_e} \mathrm{div}(\overline{\overline{F}}^T\vec{\varphi})\,dv = \sum_e \int_{\Omega_e} \vec{\varphi}\cdot\vec{S}\,dv$$

$$\mathrm{div}(\overline{\overline{F}}^T\vec{\varphi}) = \nabla\vec{\varphi}:\overline{\overline{F}}^T + \vec{\varphi}\cdot\mathrm{div}\overline{\overline{F}}$$

$$\sum_e \int_{\Omega_e} \vec{\varphi}\cdot\frac{d\vec{w}}{dt}\,dv - \sum_e \int_{\Omega_e} \nabla\vec{\varphi}:\overline{\overline{F}}^T\,dv + \sum_e \int_{\partial\Omega_e} \overline{\overline{F}}^T\vec{\varphi}\cdot\vec{n}\,ds = \sum_e \int_{\Omega_e} \vec{\varphi}\cdot\vec{S}\,dv$$

Green's formula

$$\sum_e \int_{\Omega_e} \vec{\varphi}\cdot\frac{d\vec{w}}{dt}\,dv - \sum_e \int_{\Omega_e} \nabla\vec{\varphi}:\overline{\overline{F}}^T\,dv + \sum_e \int_{\partial\Omega_e} \vec{\varphi}\cdot\overline{\overline{F}}\vec{n}\,ds = \sum_e \int_{\Omega_e} \vec{\varphi}\cdot\vec{S}\,dv$$

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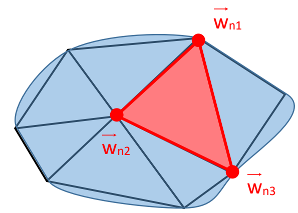
Finite Element Method

Principle

$$\frac{d\vec{w}}{dt} + div\,\bar{\bar{F}}(w) = \vec{S} \quad \text{in } \Omega$$

$$\sum_e \int_{\Omega_e} \vec{\varphi}.\frac{d\vec{w}}{dt}\, dv - \sum_e \int_{\Omega_e} \nabla\vec{\varphi} : \bar{\bar{F}}^T\, dv + \sum_e \int_{\partial\Omega_e} \vec{\varphi}.\bar{\bar{F}}\vec{n}\, ds = \sum_e \int_{\Omega_e} \vec{\varphi}.\vec{S}\, dv$$



The solutions w are approximated in each Element by a polynomial approximation

$$\vec{w}_e(x,y,z) \approx \sum_{\text{nodes of } e} N_i(x,y,z)\vec{w}_i$$

Where N_i are chosen polynomials of order 1, 2, 3, … called interpolation functions

Rmk 1: N_i =1 on node i and : Ni =0 on other nodes
Rmk 2: the approximation of w is continuous!
Rmk 3: the approximation of ∇w is NOT continuous!

$\vec{\varphi}$ Is chosen as a continuous test function $\Rightarrow \sum_e \int_{\partial\Omega_e} \vec{\varphi}.\bar{\bar{F}}\vec{n}\, ds = \sum_e \int_{\partial\Omega_e \in \partial\Omega} \vec{\varphi}.\bar{\bar{F}}\vec{n}\, ds$ Neumann condition

Using the Galerkin's rule the test functions are chosen such that: $\vec{\varphi}(x,y,z) \approx \sum_{\text{nodes of } e} N_j(x,y,z)\vec{\varphi}_j$

$$\sum_e \int_{\Omega_e} N_j\vec{\varphi}_j.\frac{d}{dt}N_i\vec{w}_i\, dv - \sum_e \int_{\Omega_e} \nabla N_j\vec{\varphi}_j : \bar{\bar{F}}^T(N_i\vec{w}_i)\, dv = \sum_e \int_{\Omega_e} N_j\vec{\varphi}_j.\vec{S}\, dv - \sum_e \int_{\partial\Omega_e \in \partial\Omega} N_j\vec{\varphi}_j.\bar{\bar{F}}\vec{n}\, ds \quad \forall\vec{\varphi}_j$$

therefore taking successively each component of the test functions equal to 1 and the others to zero, we obtain a system of equations of size : number of degree of freedom x number of nodes

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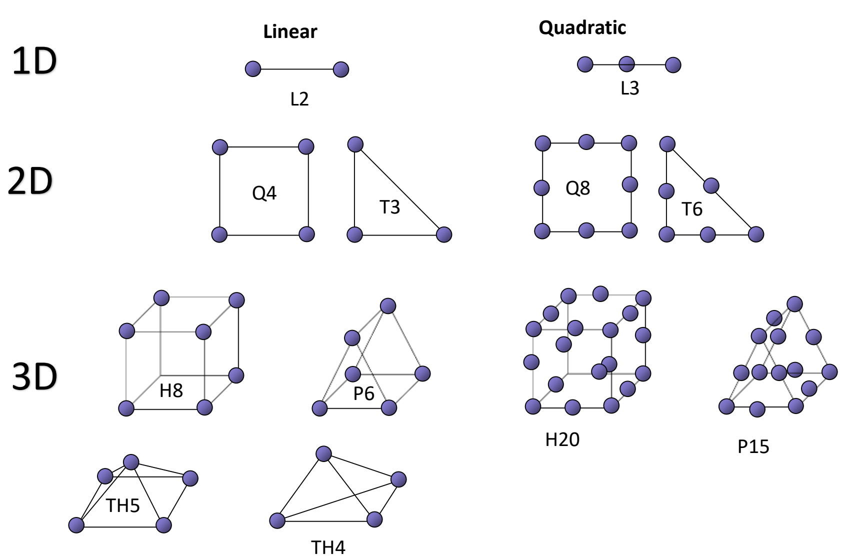
Finite Element Method

Some usual finite element types



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Finite Element Method

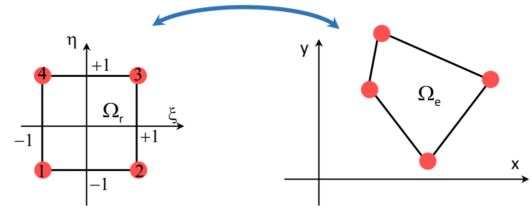
Reference Element

$\vec{x}(x,y,z) \approx \sum_{\text{nodes of } e} \bar{N}_i(x,y,z)\vec{x}_i$ $\bar{N}_i$ denote the shape functions

In the case of isoparametric element, we get $\bar{N}_i = N_i$

$$\int_{\Omega_e} \ldots \, dxdy = \int_{\Omega_r} \ldots \, \det J \, d\xi d\eta$$

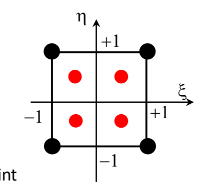
Where J denotes the Jacobian of the transformation



Reference Element   Real Element

Numerical integration

$$\int_{\Omega_r} f(\xi,\eta)\, d\xi d\eta = \sum_{\text{Gauss points}} \omega_{PG} f(\xi_{PG},\eta_{PG})$$



According to the Gaussian quadrature rule, ω_{PG} denotes the weights of each Gaussian point

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Finite Element Method

Many other notions specific to FEM to handle …

- Mesh
- Conforming mesh
- Assembly
- Lumping
- Post processing analysis
- …

Many other notions to handle in the framework of FEM

- Boundaries conditions
- Time discretisation
- Linear solver: direct, iterative
- Non-linearities, tangent operator
- Convergence
- A posteriori error
- Domain decomposition
- Adaptive mesh refinement
- HPC
- …

Finite Volume Method

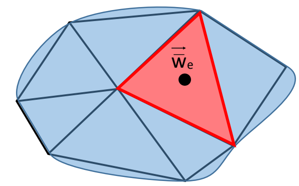
Principle

$$\frac{d\vec{w}}{dt} + \operatorname{div}\overline{\overline{F}}(w) = \vec{S} \quad \text{in } \Omega$$

$$\sum_e \int_{\Omega_e} \vec{\varphi}.\frac{d\vec{w}}{dt}\, dv - \sum_e \int_{\Omega_e} \nabla\vec{\varphi} : \overline{\overline{F}}^T\, dv + \sum_e \int_{\partial\Omega_e} \vec{\varphi}.\overline{\overline{F}}\vec{n}\, ds = \sum_e \int_{\Omega_e} \vec{\varphi}.\vec{S}\, dv$$

The solutions w are approximated in each volume by an average value

$$\vec{w}_e(x,y,z) \approx \frac{1}{|\Omega_e|}\int_{\Omega_e} \vec{w}\, dv$$



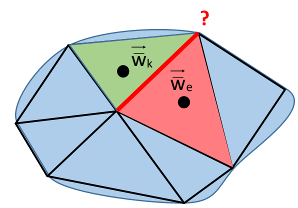
$\vec{\varphi}$ Is chosen successively as constant on volume Ω_e and null elsewhere

$$\Rightarrow \nabla\vec{\varphi} = \overline{\overline{0}}$$

$$\Rightarrow \int_{\Omega_e} \frac{d\vec{w}}{dt}\, dv + \int_{\partial\Omega_e} \overline{\overline{F}}\vec{n}\, ds = \int_{\Omega_e} \vec{S}\, dv$$

$$\Rightarrow \frac{d\vec{w}_e}{dt} + \int_{\partial\Omega_e} \overline{\overline{F}}(\vec{w}_e)\vec{n}\, ds = \overline{\overline{S}}$$

$\overline{\overline{F}}(\vec{w}_e)$ is NOT defined on $\partial\Omega_e$!



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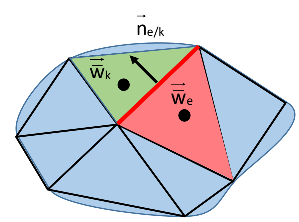
Finite Volume Method

Fluxes computation

$$\int_{\partial\Omega_e} \overline{\overline{F}}\vec{n}\, ds \approx \sum_{edges} |edge|\, f\left(\vec{w}_e, \vec{w}_k, \vec{n}_{e/k}\right)$$

- Centered
- Roe
- Lax-Friedrichs
- Engquist-Osher
- Lax-Wendroff
- Godunov : exact Riemann solver
- …



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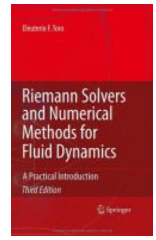
Finite Volume Method

Many other notions specific to FVM to handle …

- Structured Mesh, unstructured mesh
- Second order reconstruction (MUSCL)
- Slope limiters
- Post processing analysis
- …



Many other notions to handle in the framework of FVM

- Boundaries conditions
- Time discretisation
- Linear solver: direct, iterative, multigrid
- Interface sharpening
- Numerical diffusion
- Convergence
- A posteriori error
- Domain decomposition, parallel computing
- Adaptive mesh refinement
- HPC
- …

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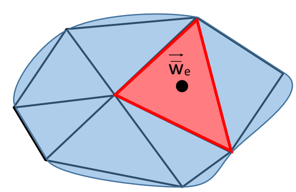
Galerkin Discontinuous Finite Element Method

Principle

$$\frac{d\vec{w}}{dt} + div\,\bar{\bar{F}}(w) = \vec{S} \quad \text{in } \Omega$$

$$\sum_e \int_{\Omega_e} \vec{\varphi}.\frac{d\vec{w}}{dt}\,dv - \sum_e \int_{\Omega_e} \nabla\vec{\varphi} : \bar{\bar{F}}^T\,dv + \sum_e \int_{\partial\Omega_e} \vec{\varphi}.\bar{\bar{F}}\vec{n}\,ds = \sum_e \int_{\Omega_e} \vec{\varphi}.\vec{S}\,dv$$



The solutions w are approximated in each Element by a polynomial approximation of order p

$$\vec{w}_e(x,y,z) \approx \sum_{i,j,k=0}^{i+j+k\le p} \vec{a}_{ijk} x^i y^j z^k$$

Rmk 1: the approximation of w NOT is continuous!
Rmk 2: We present a monomial approach, easier to implement, but harder to physically understand. A nodal approximation is still possible

$\vec{\varphi}$ Is chosen successively as a DIScontinuous test function on element Ω_e and null elsewhere

Using the Galerkin's rule the test functions are also chosen such that: $\vec{\varphi}_e(x,y,z) \approx \sum_{i,j,k=0}^{i+j+k\le p} \vec{\varphi}_{ijk} x^i y^j z^k$

$$\int_{\Omega_e} \vec{\varphi}_e.\frac{d\vec{w}_e}{dt}\,dv - \int_{\Omega_e} \nabla\vec{\varphi}_e : \bar{\bar{F}}^T\,dv + \int_{\partial\Omega_e} \vec{\varphi}_e.\bar{\bar{F}}\vec{n}\,ds = \int_{\Omega_e} \vec{\varphi}_e.\vec{S}\,dv \quad \forall \vec{\varphi}_{ijk}$$

therefore taking successively each component of the test functions equal to 1 and the others to zero, we obtain a system of equations of size : number of degree of freedom x number of coefficient of the polynomial approximation

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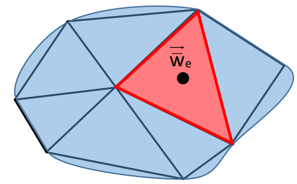
Galerkin Discontinuous Finite Element Method

Fluxes Computation

$$\int_{\Omega_e} \vec{\varphi}_e \cdot \frac{d\vec{w}_e}{dt}\, dv - \int_{\Omega_e} \nabla\vec{\varphi}_e : \bar{\bar{F}}^T\, dv + \int_{\partial\Omega_e} \vec{\varphi}_e \cdot \bar{\bar{F}}\vec{n}\, ds = \int_{\Omega_e} \vec{\varphi}_e \cdot \vec{S}\, dv \quad \forall \vec{\varphi}_{ijk}$$



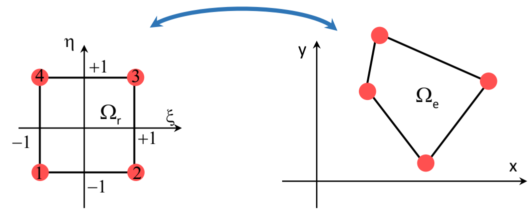
One more time, the fluxes are not defined at the interfaces between elements. An approximation has to be chosen in order to handle with the jump across the interface

Reference Element

$$\vec{x}(x,y,z) \approx \sum_{\text{nodes of e}} \bar{N}_i(x,y,z)\vec{x}_i$$

$$\int_{\Omega_e} \dots\, dxdy = \int_{\Omega_r} \dots\, \det J\, d\xi d\eta$$

Here we use a nodal approximation, often of order 1

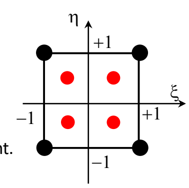


Reference Element

Real Element

Numerical integration

$$\int_{\Omega_r} f(\xi,\eta)\, d\xi d\eta = \sum_{\text{Gauss points}} \omega_{PG} f(\xi_{PG},\eta_{PG})$$



According to the Gaussian quadrature rule, ω_{PG} denotes the weights of each Gaussian point. The number of Gaussian points is adapted to the order of the polynomial approximation.

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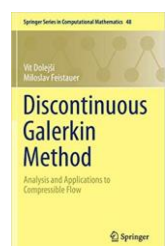
Galerkin Discontinuous Finite Element Method

Many other notions specific to GD to handle …

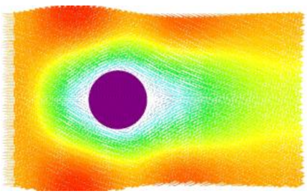
- Mesh, interface meshing
- Non conforming mesh
- Assembly
- Lumping
- Penalization
- Hp adaptation
- Post processing analysis
- …



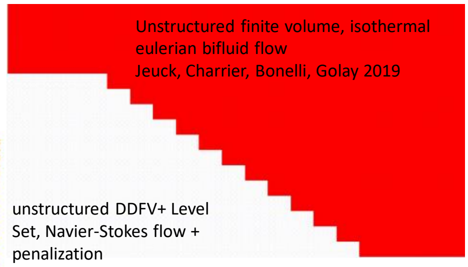
Many other notions to handle in the framework of GD

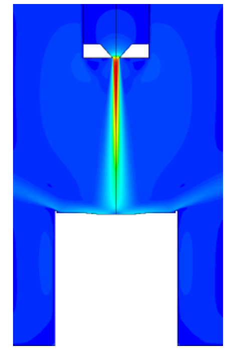
- Boundaries conditions
- Time discretisation
- Linear solver: direct, iterative
- Non-linearities, tangent operator
- Convergence
- A posteriori error
- Domain decomposition
- Adaptive mesh refinement
- HPC
- …

To be continued ……..



unstructured DDFV+ Level Set, Navier-Stokes flow + penalization
Lakhlili, Bonelli, Golay, Galusinski, 2015





Finite volume (Fluent), Navier-Stokes flow+Turbulence+ remeshing
Mercier, Bonelli, Anselmet, 2013